Journal of Approximation Theory **95**, 264–279 (1998)
Article No. AT973218

Error Estimates and Convergence Rates for Variational Hermite Interpolation

Zuhua Luo and Jeremy Levesley

Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, England

Communicated by Robert Schaback

Received February 20, 1997; accepted October 29, 1997

This paper considers the variational problem of Hermite interpolation and its error bounds. The optimal Hermite interpolant, which minimises the semi-norm of the reproducing kernel Hilbert space C_h determined by given r-CPD$_m$ function h, is just the h-spline Hermite interpolant. The results on error estimation and convergence rate of the h-spline interpolant generalise those of W. R. Madych and S. A. Nelson (1988, *Approx. Theory Appl.* **4**, 77–89; 1990, *Math. Comp.* **54**, 211–230), Z. Wu and R. Schabach (1993, *IMA J. Numer. Anal.* **13**, 13–27), and W. Light and H. Wayne (1994, *in* "Approximation Theory, Wavelets and Applications" (S. P. Singh, Ed.), pp. 215–246, Kluwer Academic, Dordrecht/Norwell, MA) to the case of Hermite interpolation. © 1998 Academic Press

1. INTRODUCTION

The classic variational approach of interpolation proposed by Duchon [4, 5], and further discussed by Meinguet [13], is to find an interpolant $u \in D^{-m}L^2(\mathbb{R}^d) = \{f\colon D^\alpha f \in L_2(\mathbb{R}^d)\ \forall\ |\alpha| = m\}$ minimising the quadratic functional

$$\|u\|_m^2 := \sum_{|\alpha| = m} c_\alpha \int_{\mathbb{R}^d} |D^\alpha u(x)|^2\, dx, \tag{1}$$

under the interpolatory constraints

$$u(x_i) = f_i, \qquad 1 \leqslant i \leqslant N, \tag{2}$$

with

$$N \geqslant d(m) := \binom{d+m-1}{d}. \tag{3}$$

0021-9045/98 $25.00

Here $m > d/2$ so that point-evaluation functionals are continuous, $x_1, ..., x_N \in \Omega$ are interpolation points, and $f_1, ..., f_N$ are data. The positive constants c_α are specified by

$$|\xi|^{2m} = \sum_{|\alpha| = m} c_\alpha \xi^{2\alpha}.$$

Denote by $\Pi_k(\mathbb{R}^d)$ the space of polynomials of degree less than or equal to k in d variables, and let $n = \dim(\Pi_k(\mathbb{R}^d))$. furthermore, let the polyharmonic splines h be defined by

$$h(r) = \begin{cases} r^{2m-d}, & d \text{ odd}, \\ r^{2m-d} \log r, & d \text{ even}. \end{cases} \tag{4}$$

Then, the solution of the above constrained interpolation problem is just the h-spline interpolant; i.e., the interpolant is of the form

$$s(x) := \sum_{i=1}^{N} a_i h(\|x - x_i\|) + \sum_{|\alpha| < m} b_\alpha x^\alpha, \tag{5}$$

where

$$\sum_{i=1}^{N} a_i p(x_i) = 0 \qquad \forall p \in \Pi_{m-1}(\mathbb{R}^d). \tag{6}$$

In [11, 12] Madych and Nelson describe a variational approach for interpolation using a fixed conditionally positive definite (CPD) function. Unlike Duchon's approach, they begin with a continuous CPD function h and construct a semi-Hilbert space C_h based on h. Then using the reproducing kernel Hilbert space theory they establish pointwise error estimates as the interpolation points become dense in Ω. Later, Wu and Schabach [18] and Light and Wayne [10] exploit Kriging methods and representer theory, respectively, to arrive at similar error estimates. Here we modify and extend the method from [11, 12] to examine convergence rates for Hermite interpolation, again using CPD functions. One can also use the Kriging method or representer theory to attack the same problem. It should be pointed out that all these approaches, coming from different perspectives though they do, conform with the variational spline theory presented in its general form by Atteia [1], Bezhaev and Vasilenko [2], or Laurent [8]. This is also true of the theory presented in this paper.

To describe the Hermite problem we introduce the space $\mathbf{C}^r$ composed of r-times continuously differentiable functions in $\mathbb{R}^d$, and its dual space $\mathscr{E}'_r := \{g \colon g$ is a compactly supported distribution on $\mathbb{R}^d$ of degree at most $r\}$. Here r is the largest integer number such that the CPD function $h \in \mathbf{C}^{2r}(\mathbb{R}^d)$. For $L \in \mathscr{E}'_r$ and $f \in \mathbf{C}^r$, let

$$(L, f)$$

denote the action of distribution L on the test function f. In Section 2 we will construct the Hilbert space C_h based on h, possessing semi-norm $|\cdot|_h$, in which our analysis occurs. Given a set of distributions $\Lambda := \{L_1, ..., L_N\} \subset \mathscr{E}'_r$ and data $f_1, ..., f_N$, let $V_\Lambda = \{u \in C_h: (L_i, u) = f_i, i = 1, ..., N\}$. We wish to find $u \in C_h$ such that

$$|u|_h = \inf_{v \in V_\Lambda} |v|_h.$$

Such a solution u is called the optimal Hermite interpolant. We assume that the set Λ contains unisolvent subset over Π_{m-1}, which is to say that if $(L, p) = 0$, $p \in \Pi_{m-1}$, for all $L \in \Lambda$, then $p = 0$.

With reference to the general framework [1], we show that the h-spline Hermite interpolant

$$s(x) := \sum_{i=1}^{N} a_i L_i * h(x) + \sum_{|\alpha| < m} b_\alpha x^\alpha$$

satisfying

$$(L_i, s) = f_i, \qquad i = 1, ..., N,$$

and

$$\sum_{i=1}^{N} a_i (L_i, p) = 0 \qquad \text{for} \quad p \in \Pi_{m-1}(\mathbb{R}^d)$$

is the unique optimal Hermite interpolant from C_h. Here the convolution $L_i * h$ of distribution L_i with function $h \in \mathbf{C}^{2r}(\mathbb{R}^d)$ is defined as usual, i.e., $(L_i * h)(x) = (L_i, h(x - \cdot))$. Using this fact we close Section 2 with a bound on $|(M, f - s)|$ for a given $M \in \mathscr{E}'_r$ and $f \in C_h$. The main result of the paper is in Section 3, where we assume that

$$Y_\alpha = \{ y \in \mathbb{R}^d \mid D^\alpha \delta_y \in \Lambda \}$$

satisfies

$$\operatorname{dist}(Y_\alpha, \Omega) := \sup_{x \in \Omega} \min_{y \in Y_\alpha} |x - y| \leqslant \rho,$$

for some given α such that $|\alpha| \leqslant r$. Then, if $M = D^\beta \delta_y$ for some $\beta \in \mathbb{Z}^d$ such that $|\beta| \leqslant r$, $\beta \geqslant \alpha$, and $y \in \Omega$, we obtain the pointwise convergence rate of the error bound $|(M, f-s)|$ on Ω in terms of the density measure ρ of Y_α on Ω,

$$|(M, f-s)| \leqslant C\rho^{r-|\beta|},$$

where C is a constant, depending on f, but independent of ρ. As is shown in Duchon [4], this can bee extended to a uniform bound on Ω if Ω satisfies, for example, a cone condition.

The solvability of the h-spline Hermite interpolation has been studied by several authors such as Wu [17], Sun [16], Jetter *et al.* [7], and Narcowich and Ward [14]. They get certain sufficient conditions to guarantee the well-poisedness of Hermite interpolation. One such sufficient condition is that h be strictly r-CPD$_m$. Let $\mathscr{E}'_{m,r} = \{L \in \mathscr{E}'_r : (L, p) = 0, \forall p \in \Pi_{m-1}\}$.

DEFINITION 1. A function h is called r-CPD$_m$ if $h \in \mathbf{C}^{2r}(\mathbb{R}^d)$ and $(L, L*h) \geqslant 0$, whenever $L \in \mathscr{E}'_{m,r}$. If the inequality holds strictly for all nonzero members of $\mathscr{E}'_{m,r}$ then h is called strictly r-CPD$_m$.

We point out that $(L, L*h)$ is well defined if $L \in \mathscr{E}'_r$ and $h \in \mathbf{C}^{2r}(\mathbb{R}^d)$. When L is pointwise supported, $L = D^\alpha \delta_{x_0}$, for example, then $L*h(x) = D^\alpha_y h(x-y)|_{y=x_0}$. In such cases, we write $L*h(x) = Lh(x-\cdot)$.

LEMMA 1. *Let h have positive generalized Fourier transform $\hat{h}$ on $\mathbb{R}^d/\{0\}$ satisfying*

$$\int_{|\xi|<1} |\xi|^{2m}\, \hat{h}(\xi)\, d\xi < \infty, \qquad \int_{|\xi|>1} |\xi|^{2r}\, \hat{h}(\xi)\, d\xi < \infty. \tag{7}$$

Then for any $L \in \mathscr{E}'_{m,r}$,

$$|(L, e^{i\cdot\xi})|^2 = \mathscr{O}(|\xi|^{2m}), \qquad \xi \to 0, \qquad |(L, e^{i\cdot\xi})|^2 = \mathscr{O}(|\xi|^{2r}), \qquad \xi \to \infty. \tag{8}$$

Moreover, h is strictly r-CPD$_m$.

Proof. That Eqs. (8) are true is easily deduced from the fact that L annihilates Π_{m-1} and by the definition of the order of distributions; see Donoghue [3]. Since h satisfies

$$\int_{|\xi|<1} |\xi|^{2m}\, \hat{h}(\xi)\, d\xi < \infty,$$

and

$$\int_{|\xi|>1} |\xi|^{2r} \hat{h}(\xi)\, d\xi < \infty,$$

the integral

$$\int_{\mathbb{R}^d} \hat{h}(\xi)\, |(L, e^{i\cdot\xi})|^2\, d\xi$$

is convergent. Thus using elementary properties of distributional Fourier transforms, we have

$$\begin{aligned}(L, L * h) &= (\hat{L}, \hat{L}\hat{h}) \\ &= \int_{\mathbb{R}^d} \overline{\hat{h}(\xi)\, \hat{L}(\xi)}\, \hat{L}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^d} \hat{h}(\xi)\, |(L, e^{i\cdot\xi})|^2\, d\xi \\ &> 0, \end{aligned} \tag{9}$$

where we have used the formula $\hat{L}(\xi) = (L, e^{i\cdot\xi})$. Thus h is strictly r-CPD$_m$. ∎

According to Lemma 1, the Gaussian kernel $e^{-|x|^2}$ is strictly r-CPD$_m$ for $m \geqslant 0$ and $r < \infty$; the thin plate splines as given in (4), for $r < m - d/2$; the multiquadrics $\sqrt{1+|x|^2}$ for $m \geqslant 1$ and $r < \infty$; and the inverse multiquadric $1/\sqrt{1+|x|^2}$ for $m \geqslant 0$ and $r < \infty$. In this paper, we will always assume that h satisfies conditions (7).

2. ERROR ESTIMATES FOR h-SPLINE INTERPOLATION

For $L_1, L_2 \in \mathscr{E}'_{m,r}$, define

$$\langle L_1, L_2 \rangle := (L_1, L_2 * h).$$

Then, as h is strictly r-CPD$_m$, $\langle \cdot, \cdot \rangle$ is an inner product on $\mathscr{E}'_{m,r}$. Let H be the Hilbert space completion of $\mathscr{E}'_{m,r}$ with the norm $\|\cdot\|_H$ inherited from $\langle \cdot, \cdot \rangle^{1/2}$. Define the map T by

$$T\colon L \in H \to T(L) := L * h, \tag{10}$$

and the space

$$D_h := \{T(L) = L * h \mid L \in H\}.$$

The mapping T is one to one because h is strictly r-CPD$_m$. Endow D_h with the inner product

$$(f, g)_h := (T^{-1}(f), T^{-1}(g))_H. \tag{11}$$

Then $\|\cdot\|_h$ is a norm and $(D_h, \|\cdot\|_h)$ is a Hilbert space. By the definition of D_h we know that D_h is orthogonal to polynomial subspace Π_{m-1}. Let

$$C_h = D_h \oplus \Pi_{m-1}$$

and endow C_h with the semi-inner product

$$(f, g)_h := (f_1, g_1)_h,$$

when $f = f_1 + f_2$, $g = g_1 + g_2$ with $f_1, g_1 \in D_h$ and $f_2, g_2 \in \Pi_{m-1}$. Then $(C_h, |\cdot|_h)$ is a semi-Hilbert space with a semi-norm $|\cdot|_h = (f, f)_h^{1/2}$, having the polynomial subspace Π_{m-1} as its kernel.

THEOREM 1. *Let the strictly r-CPD$_m$ kernel h be as in Lemma* 1. *Then,*

(a) *C_h is continuously imbedded in* $\mathbf{C}^r(\mathbb{R}^d)$.

(b) *if* $L \in \mathscr{E}'_{m,r}$,

$$(L, f) = (L * h, f)_h \qquad \textit{for all} \quad f \in C_h.$$

Proof. Statement (a) follows directly from the definition of C_h due to the fact that $h \in \mathbf{C}^{2r}(\mathbb{R}^d)$. Let $L \in \mathscr{E}'_{m,r}$, and let $f \in C_h$. By definition of C_h, there exists a $\gamma \in H$ such that $f := f_\gamma = \gamma * h + P_{m-1}$, where $P_{m-1} \in \Pi_{m-1}$. Then we have

$$\begin{aligned}(L, f) &= (L, f_\gamma) = (L, \gamma * h + P_{m-1})\\ &= (L, \gamma * h) = (L, \gamma)_H\\ &= (L * h, \gamma * h)_h = (L * h, f_\gamma)_h\\ &= (L * h, f)_h,\end{aligned}$$

which proves (b). ∎

As in [12] it is possible to describe more explicitly the elements of C_h. However, here we are mainly concerned with producing an error estimate. To see how we do this in detail the reader is referred to [9], but here it is more appropriate to appeal to the general setting of Atteia to indicate how the required estimate is obtained. The important ingredients of the theory are the semi-Hilbert space C_h, the reproducing kernel property (Theorem 1(b)), and the set Λ, which, by Theorem 1(a), is a set of continuous functionals from C_h into $\mathbb{R}$. For then we know that there is a unique element of C_h,

$$s(x) = \sum_{i=1}^{N} a_i L_i * h(x) + p_{m-1}(x),$$

where $p_{m-1} \in \Pi_{m-1}$, such that, for any $v \in V_\Lambda$

$$|v|_h^2 = |v - s|_h^2 + |s|_h^2. \tag{12}$$

We recall here that, given a data set $f_1, ..., f_N$, $V_\Lambda = \{v \in C_h: (L_i, v) = f_i, i = 1, ..., N\}$. An immediate consequence of the last equation is the semi-norm minimization property

$$|s|_h = \min_{v \in V_\Lambda} |v|_h.$$

Now we estimate the error $|(M, f - s)|$, where s is the h-spline Hermite interpolant to f and $M \in \mathscr{E}'_r$. For this purpose we define a distribution L by

$$L = M + \sum_{k=1}^{N} c_k L_k,$$

where $c_1, ..., c_N$ are so chosen that $L \in \mathscr{E}'_{m,r}$. This is always possible because Λ is linearly independent over Π_{m-1}. In fact, there exists a subset, say $\{L_1, ..., L_n\} \subset \Lambda$, where $n = \dim \Pi_{m-1}$, having the same property. Furthermore, there exists a basis of polynomials $p_1, ..., p_n \in \Pi_{m-1}$ which form a biorthonormal basis $\{L_1, ..., L_n\}$, i.e., $(L_i, p_j) = \delta_{i,j}$, $i, j = 1, ..., n$. Let

$$\begin{cases} c_i = -(M, p_i), & i = 1, ..., n; \\ c_i = 0, & i = n+1, ..., N. \end{cases} \tag{13}$$

Then the functional $L = M + \sum_{k=0}^{N} c_k L_k$ will annihilate Π_{m-1}. Moreover

$$\begin{aligned} (L, f - s) &= M + \sum_{i=1}^{N} c_i (L_i, f - s) \\ &= (M, f - s), \end{aligned}$$

as $(L_i, s) = (L_i, f)$ for $i = 1, ..., N$. Thus by (b) of Theorem 1,

$$\begin{aligned}|(M, f-s)| &= |(L, f-s)| \\ &= (L * h, f-s)_h \\ &\leqslant \|L * h\|_h \, |f-s|_h \\ &\leqslant \|L * h\|_h \, |f|_h,\end{aligned}$$

using (12). Hence, in view of (9), (10), and (11), we obtain

THEOREM 2. *Let h satisfy the conditions of Lemma 1. Let s be the h-spline Hermite interpolant to $f \in C_h$. For each functional $M \in \mathscr{E}'_r$, which is linearly independent of $L_1, ..., L_N$, on $\mathbf{C}^r(\mathbb{R}^d)$, let L be the functional defined by*

$$L = M + \sum_{k=1}^{N} c_k L_k,$$

where $c_1, ..., c_N$ are so chosen that $L \in \mathscr{E}'_{m,r}$. Then

$$|(M, f-s)| \leqslant C(M) \, |f|_h,$$

where

$$C(M) = \inf_{\mathbf{c} \in \mathbb{R}^N} \left\{ \left(\int_{\mathbb{R}^d} \hat{h}(\xi) \left| \hat{M}(\xi) + \sum_{j=1}^{N} c_j \widehat{L_j}(\xi) \right|^2 d\xi \right)^{1/2} : M + \sum_{j=1}^{N} c_j L_i \in \mathscr{E}'_{m,r} \right\}.$$

3. POINTWISE CONVERGENCE RATES

Throughout this section, C will be an intrinsic constant which will not necessarily be the same at each occurrence.

In [12] and [18], Madych and Nelson and Wu and Schaback, respectively, give convergence rates for h-spline Lagrange interpolation in C_h. Here we will extend their results to the case of Hermite interpolation on Ω, a domain which is nice enough that there exist positive constants K, ε_0 such that for every $0 < \varepsilon < \varepsilon_0$,

$$\Omega \subset \bigcup \{B(t, \varepsilon K) : t \in T_\varepsilon\},$$

where

$$T_\varepsilon = \{t \in \mathbb{R}^d : B(t, \varepsilon) \subset \Omega\}, \qquad B(t, \varepsilon) = \{x \in \mathbb{R}^d : |x - t| \leqslant \varepsilon\}.$$

If Ω satisfies the cone condition, then the above requirements will naturally be met; see [5]. Let $\{L_1, ..., L_N\}$ be the interpolating functionals with supporting set X. Let s be the h-spline Hermite interpolant to f. For each $\beta \in \mathbb{Z}^d$, $0 \leqslant |\beta| \leqslant r$, and $x \in \Omega$, we want to get the convergence rate of $|D^\beta(s-f)(x)|$ on Ω, as the coverage of X to the domain Ω improves.

In the case of Lagrange interpolation, the support of point-evaluation functionals is required to become dense on Ω. In the case of Hermite interpolation we require that for certain multiinteger $\alpha \in \mathbb{Z}^d$, $|\alpha| \leqslant r$, the subset Y_α of X defined by

$$Y_\alpha = \{ y \in \mathbb{R}^d \mid D^\alpha \delta_y \in \{L_1, ..., L_N\}\}$$

satisfy

$$\operatorname{dist}(Y_\alpha, \Omega) \leqslant C\rho,$$

and we estimate the pointwise error $|D^\beta(s-f)(x)|$ in terms of the parameter ρ, when $\beta \in \mathbb{Z}^d$ such that $|\beta| \leqslant r$ and $\beta \geqslant \alpha$.

Now for every integer $l \geqslant \max\{m, r\}$ and fixed α, let

$$\Pi_\alpha = \{ p \in \Pi_{l-1} \mid D^\alpha(p) = 0\} \tag{14}$$

and let $\Pi_l^\alpha := \Pi_{l-1}(\mathbb{R}^d)/\Pi_\alpha(\mathbb{R}^d)$ be the quotient set in which the element 0 is identified with any $p \in \Pi_\alpha(\mathbb{R}^d)$. Let

$$n := \dim \Pi_l^\alpha = \dim \Pi_{l-1}(\mathbb{R}^d) - \dim \Pi_\alpha.$$

Note that we can always select a set of n functionals $\{L_{\mathbf{a}(i)} = D^\alpha \delta_{\mathbf{a}(i)};\ i = 1, ..., n\}$, where $\mathbf{a}(i) \in \mathbb{R}^d$, such that these functionals are linearly independent of Π_l^α. Hence for any set of n data $\{\beta_1, ..., \beta_n\}$, there exists a unique polynomial $p \in \Pi_l^\alpha$ such that

$$(L_{\mathbf{a}(i)}, p) = \beta_i, \qquad i = 1, ..., n.$$

Let $\{P^{\mathbf{a}(i)};\ i = 1, ..., n\} \subset \Pi_l^\alpha$ satisfy

$$(L_{\mathbf{a}(i)}, P^{\mathbf{a}(j)}) = \delta_{i,j}, \qquad i, j = 1, ..., n.$$

Let $\mathbf{B}(\lambda)$ be a neighbourhood of points $\{\mathbf{a}(i) \in \mathbb{R}^d;\ i = 1, ..., n\}$ for some constant $\lambda > 0$, i.e.,

$$\mathbf{B}(\lambda) = \bigoplus_{i=1}^{n} B(\mathbf{a}(i), \lambda).$$

Clearly, $\mathbf{b} = (\mathbf{b}(1), ..., \mathbf{b}(n)) \in \mathbf{B}(\lambda)$ if and only if

$$|\mathbf{b}(i) - \mathbf{a}(i)| < \lambda, \qquad i = 1, ..., n.$$

Since $\{L_{\mathbf{a}(i)}; i=1, ..., n\}$ are functionals linearly independent of Π_l^α, by continuity, we can choose $\lambda > 0$ such that for any $\mathbf{b} \in \mathbf{B}(\lambda)$, $\{L_{\mathbf{b}(i)}; i=1, ..., n\}$ are also functionals linearly independent of Π_l^α. Now choose a constant $R \geqslant \max\{1, \varepsilon_0, \lambda\}$ such that

$$B(\mathbf{a}(i), \lambda) \subset B(0, R), \qquad i=1, ..., n.$$

For $\rho = \rho(\Omega, Y) < \varepsilon_0 \lambda / R$, set $\varepsilon = R\rho\lambda^{-1} = R\underline{\rho}$, where $\underline{\rho} = \rho\lambda^{-1}$. For a fixed point $x \in \Omega$, we choose $t \in B(x, \varepsilon) \cap T_\varepsilon$ and consider the set of points $\{t + \underline{\rho}\mathbf{a}(i)\}_1^n$. It is easy to verify that the balls $B(t + \underline{\rho}\mathbf{a}(i), \rho)$ are contained in $B(t, R\underline{\rho}) = B(t, \varepsilon) \subset \Omega$. Since ρ is the measure of how closely Y_α covers Ω, there exist, say, $\mathbf{x}(i) \in Y_\alpha$ such that

$$\mathbf{x}(i) \in B(t + \underline{\rho}\mathbf{a}(i), \rho), \qquad i=1, ..., n, \tag{15}$$

and moreover

$$|x - \mathbf{x}(i)| \leqslant |x - t| + |t - \mathbf{x}(i)| \leqslant \varepsilon + \underline{\rho} R = 2R\lambda^{-1}\rho, \tag{16}$$

where x is the fixed point in Ω. Now let

$$\mathbf{b}(i) = \frac{\mathbf{x}(i) - t}{\underline{\rho}}, \qquad i=1, ..., n; \tag{17}$$

then

$$\mathbf{b}(i) \in B(\mathbf{a}(i), \rho) \subset B(\mathbf{a}(i), \lambda), \qquad i=1, ..., n.$$

Hence the functionals $L_{\mathbf{b}(i)}$ are linearly independent of $\Pi_l(\mathbb{R}^d)$, and we can construct the corresponding polynomials $P^{\mathbf{b}(i)} \in \Pi_l^\alpha$ such that

$$(L_{\mathbf{b}(i)}, P^{\mathbf{b}(j)}) = \delta_{i,j}, \qquad i=1, ..., n. \tag{18}$$

Lemma 2. *Let*

$$L := \Gamma - \sum_{i=1}^{n} u_i(x) L_{\mathbf{x}(i)},$$

where $\Gamma = D^\beta \delta_x$ *with* $\beta \geqslant \alpha$, $x \in \Omega$, *and*

$$u_i(x) = \underline{\rho}^{k_0 - k_1} (D^\beta P^{\mathbf{b}(i)})(y),$$

where $k_0 = |\alpha|$, $k_1 = |\beta|$, *and*

$$y = (x - t)/\underline{\rho} \in B(0, R).$$

Then $L \in \mathscr{E}'_{l,r}$.

Proof. We need only prove that $(L, p) = 0$ for $p \in \Pi_l^\alpha$, as Γ and $L_{\mathbf{x}(i)} = D^\alpha \delta_{\mathbf{x}(i)}$ annihilate Π_α. For any such p, we write

$$p(\cdot) = q\left(\frac{\cdot - t}{\underline{\rho}}\right) = \sum_{i=1}^{n} c_i P^{\mathbf{b}(i)}\left(\frac{\cdot - t}{\underline{\rho}}\right),$$

since $P^{\mathbf{b}(i)}$, $i = 1, ..., n$, are a basis for Π_l^α. Thus

$$\begin{aligned}
(L, p) &= (\Gamma, p) - \sum_{i=1}^{n} u_i(x)(L_{\mathbf{x}(i)}, p) \\
&= (\Gamma, p) - \sum_{i=1}^{n} u_i(x)\left(L_{\mathbf{x}(i)}, \sum_{j=1}^{n} c_j P^{\mathbf{b}(j)}\left(\frac{\cdot - t}{\underline{\rho}}\right)\right) \\
&= (\Gamma, p) - \sum_{j=1}^{n} c_j \sum_{i=1}^{n} u_i(x)\left(L_{\mathbf{x}(i)}, P^{\mathbf{b}(j)}\left(\frac{\cdot - t}{\underline{\rho}}\right)\right) \\
&= (\Gamma, p) - \sum_{j=1}^{n} c_j \sum_{i=1}^{n} u_i(x)(L_{\mathbf{b}(i)}, P\mathbf{b}(j))\, \underline{\rho}^{-k_0} \\
&= (\Gamma, p) - \sum_{j=1}^{n} c_j \sum_{i=1}^{n} \underline{\rho}^{k_0 - k_1}(D^\beta P^{\mathbf{b}(i)})(y)\, \delta_{i,j} \underline{\rho}^{-k_0} \\
&= (\Gamma, p) - \sum_{j=1}^{n} c_j \underline{\rho}^{-k_1}(D^\beta P^{\mathbf{b}(i)})(y) \\
&= (\Gamma, p) - \sum_{j=1}^{n} c_j (D^\beta P^{\mathbf{b}(i)})\left(\frac{x - t}{\underline{\rho}}\right) \\
&= (\Gamma, p) - \left(D^\beta \delta_x, \sum_{j=1}^{n} c_j P^{\mathbf{b}(i)}\left(\frac{x - t}{\underline{\rho}}\right)\right) \\
&= (\Gamma, p) - (\Gamma, p) = 0. \quad \blacksquare
\end{aligned}$$

Since $\{P^{\mathbf{b}(i)}(\xi)\}_1^n$ are polynomials, together with their derivatives up to r they are uniformly bounded on the bounded domain $\{|\xi| \leqslant R, \mathbf{b} \in \mathbf{B}(\lambda)\}$; i.e., there exists a constant $C > 0$, which only depends on λ and R, such that

$$\sup\left\{\sum_{i=1}^{n} |D^\alpha P^{\mathbf{b}(i)}(\xi)| : |\xi| \leqslant R, \mathbf{b} \in \mathbf{B}(\lambda), |\alpha| \leqslant r\right\} < C.$$

Thus for $u_i(x)$ defined in Lemma 2, we have

$$|u_i(x)| \leqslant C\underline{\rho}^{k_0-k_1}, \qquad i=1, ..., n, \tag{19}$$

for any $x \in \Omega$, as $|y| = |(x-t)/\underline{\rho}| \leqslant \varepsilon/\underline{\rho} = R$.

LEMMA 3. *Let L be defined as in Lemma 2. Then,*

$$|(L, e^{i\cdot\xi})| \leqslant C \begin{cases} \underline{\rho}^{l-k_1} |\xi|^l, & \underline{\rho}\, |\xi| \leqslant 1, \\ \underline{\rho}^{k_0-k_1} |\xi|^{k_0} + |\xi|^{k_1}, & \underline{\rho}\, |\xi| \geqslant 1. \end{cases}$$

Proof. Since for $t \in \mathbb{R}$,

$$e^t = P_{l-1}(t) + t^l R_l(t), \qquad \text{with} \quad |R_l(t)| \leqslant e^{|t|},$$

where $P_{l-1} \in \Pi_{l-1}(\mathbb{R})$. Noting that L annihilates polynomials of degree $l-1$, for $\underline{\rho}\, |\xi| \leqslant 1$, we have

$$\begin{aligned}
|(L, e^{i\cdot\xi})| &= |(L, e^{i(\cdot - x)\,\xi})| \\
&= |(L, P_{l-1}(i(\cdot - x)\,\xi) + (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi))| \\
&= |(L, (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi))| \\
&= \Bigg|(\Gamma, (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi)) \\
&\qquad - \sum_{j=1}^n u_j(x)(L_{\mathbf{x}(j)}, (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi))\Bigg| \\
&\overset{*}{=} \left|\sum_{j=1}^n u_j(x)(L_{\mathbf{x}(j)}, (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi))\right| \\
&\leqslant \left(\sum_{j=1}^n |u_j(x)|\right) \max_{1 \leqslant j \leqslant n} \{|(L_{\mathbf{x}(j)}, (i(\cdot - x)\,\xi)^l\, R_l(i(\cdot - x)\,\xi))|\} \\
&\leqslant C\underline{\rho}^{k_0-k_1} \max_{1 \leqslant j \leqslant n} |\mathbf{x}(j) - x|^{l-k_0}\, |\xi|^l \cdot e^{|\mathbf{x}(j)-x|\,|\xi|} \\
&\leqslant C\underline{\rho}^{l-k_1}\, |\xi|^l.
\end{aligned}$$

The equality of $(*)$ is valid because Γ is supported at point x. When $\underline{\rho}\, |\xi| > 1$, since the functionals Γ and $L_{\mathbf{x}(i)}$ are distributions of order k_1 and k_0, respectively, it follows that

$$|(L, e^{i\cdot\xi})| \leqslant |(\Gamma, e^{i\cdot\xi})| + \sum_{j=1}^{n} |u_j(x)|\,|(L_{\mathbf{x}(i)}, e^{i\cdot\xi})|$$
$$\leqslant |C\,|\xi|^{k_1} + C\underline{\rho}^{k_0-k_1}\,|\xi|^{k_0}|,$$

by (19). ∎

Now we state the main theorem of this section.

THEOREM 3. *Let $\Lambda = \{L_1, ..., L_N\} \subset \mathscr{E}'_r$, $r \in \mathbb{N}$, have support X, and let h satisfy the assumptions of Lemma 1. Let $f \in C_h$ and s be the h-spline Hermite interpolant to f, i.e.,*

$$s(x) = \sum_{i=1}^{n} c_i L_i * h(x) + \sum_{|\beta| \leqslant m-1} b_\beta x^\beta$$
$$(L_i, s) = (L_i, f), \qquad i = 1, ..., N,$$

where the coefficients $c_1, ..., c_N$ satisfy

$$\sum_{i=1}^{N} c_i (L_i, p) = 0, \qquad \forall p \in \Pi_{m-1}(\mathbb{R}^d).$$

If the subset Y_α of X defined by

$$Y_\alpha = \{ y \in \mathbb{R}^d \mid D^\alpha \delta_y \in \Lambda \}$$

satisfies

$$\operatorname{dist}(Y_\alpha, \Omega) \leqslant \rho,$$

then for any $\beta \geqslant \alpha$ with $|\beta| \leqslant r$, and fixed $x \in \Omega$,

$$|D^\beta (f-s)(x)| \leqslant C\,|f|_h\,\rho^{r-|\beta|}, \tag{20}$$

for some constant $C > 0$ independent of ρ and $x \in \Omega$.

Proof. Let $\Gamma = D^\beta \delta_x$ and $k_1 = |\beta|$. According to Theorem 2,

$$|D^\beta (f-s)(x)| = |\Gamma(f-s)| \leqslant c(\Gamma)\,\|f\|_h, \tag{21}$$

where

$$c(\Gamma) = \inf\left\{ \left(\int_{\mathbb{R}^d} \left| \left(\Gamma + \sum_{i=1}^{N} c_i L_i, e^{i\cdot\xi} \right) \right|^2 \hat{h}(\xi)\, d\xi \right)^{1/2} \middle|\ \Gamma + \sum_{i=1}^{N} c_i L_i \in \mathscr{E}'_{m,r} \right\}.$$

Now,

$$L = \Gamma - \sum_{i=1}^{n} u_i(x)\, L_{\mathbf{x}(i)},$$

where for $1 \leqslant i \leqslant N$,

$$c_i = \begin{cases} -u_j(x), & \text{if } \quad L_i = L_{\mathbf{x}(j)} \qquad \text{for some } \quad 1 \leqslant j \leqslant n, \\ 0, & \text{otherwise}, \end{cases}$$

annihilates Π_{l-1} (hence it also annihilates Π_{m-1}). Thus

$$c(\Gamma) \leqslant \left(\int_{\mathbb{R}^d} |(L, e^{i \cdot \xi})|^2\, \hat{h}(\xi)\, d\xi \right)^{1/2}.$$

Setting $k_0 = |\alpha|$ and $k_1 = |\beta|$, by Lemma 3, we have

$$\begin{aligned}
&\int_{\underline{\rho}\,|\xi| > 1} |(L, e^{i \cdot \xi})|^2\, \hat{h}(\xi)\, d\xi \\
&\qquad \leqslant \int_{\underline{\rho}\,|\xi| > 1} C(|\xi|^{2k_1} + \underline{\rho}^{2k_0 - k_1}\, |\xi|^{k_0})\, \hat{h}(\xi)\, d\xi \\
&\qquad \leqslant C \left\{ \int_{\underline{\rho}\,|\xi| > 1} |\xi|^{-(2r - 2k_1)}\, |\xi|^{2r}\, \hat{h}(\xi)\, d\xi \right. \\
&\qquad\qquad \left. + \int_{\underline{\rho}\,|\xi| > 1} \underline{\rho}^{2k_0 - k_1}\, |\xi|^{-(2r - 2k_0)}\, |\xi|^{2r}\, \hat{h}(\xi)\, d\xi \right\} \\
&\qquad \leqslant C(\underline{\rho}^{2r - 2k_1} + \underline{\rho}^{2k_0 - 2k_1} \rho^{2r - 2k_0}) \int_{\underline{\rho}\,|\xi| > 1} |\xi|^{2r}\, \hat{h}(\xi)\, d\xi \\
&\qquad \leqslant C \underline{\rho}^{2r - 2k_1},
\end{aligned}$$

as

$$\int_{\underline{\rho}\,|\xi| > 1} |\xi|^{2r}\, \hat{h}(\xi)\, d\xi < \infty.$$

On the other hand, again by Lemma 3,

$$
\begin{aligned}
&\int_{\underline{\rho}\,|\xi| \leqslant 1} |(L, e^{i\cdot\xi})|^2 \, \hat{h}(\xi)\, d\xi \\
&\quad \leqslant C\underline{\rho}^{2l-2k_1} \int_{\underline{\rho}\,|\xi| \leqslant 1} |\xi|^{2l} \, \hat{h}(\xi)\, d\xi \\
&\quad = C\underline{\rho}^{2l-2k_1} \left\{ \int_{|\xi|<1} |\xi|^{2l} \, \hat{h}(\xi)\, d\xi + \int_{1 \leqslant \|\xi\| \leqslant \underline{\rho}^{-1}} |\xi|^{2l} \, \hat{h}(\xi)\, d\xi \right\} \\
&\quad \leqslant C\underline{\rho}^{2l-2k_1} \left\{ \mathcal{O}(1) + \underline{\rho}^{2r-2l} \int_{1 \leqslant |\xi| \leqslant \underline{\rho}^{-1}} |\xi|^{2r} \, |\xi|^{2l-2r} \, \underline{\rho}^{2l-2r} \hat{h}(\xi)\, d\xi \right\} \\
&\quad \leqslant C\underline{\rho}^{2l-2k_1} \left\{ \mathcal{O}(1) + \underline{\rho}^{2r-2l} \int_{1 \leqslant |\xi| \leqslant \underline{\rho}^{-1}} |\xi|^{2r} \, \hat{h}(\xi)\, d\xi \right\} \\
&\quad \leqslant C\underline{\rho}^{2l-2k_1} (\mathcal{O}(1) + \mathcal{O}(\underline{\rho}^{2r-2l})) \\
&\quad = C\underline{\rho}^{2r-2k_1}.
\end{aligned}
$$

Combination of the last two inequalities tells us that

$$C(\Gamma) \leqslant C\rho^{r-k_1},$$

which, in view of (21), gives us the required result. ∎

ACKNOWLEDGMENTS

We are very grateful to the referees for valuable suggestions in the modification of the final version of the paper and for pointing out the connection of the theory with these variational spline theories of Atteia [1], Bezhaev and Vasilenko [2], and Laurent [8].

REFERENCES

1. M. Atteia, "Hilbertian Kernels: Spline Functions," Studies in Computational Mathematics IV, North-Holland, Amsterdam, 1994.
2. A. Yu. Bezhaev and V. A. Vasilenko, "Variational Spline Theory," *Bull. Novosibirsk Comput. Center*, Special Issue No. 3 (1993).
3. W. F. Donoghue, "Distributions and Fouriertransforms," Academic Press, New York, 1969.
4. J. Duchon, Splines minimising rotation-invariant semi-norms in the Sobolev spaces, *in* "Constructive Theory of Functions of Several Variables" (W. Schempp and K. Zeller, Eds.), Lecture Notes in Mathematics, Vol. 571, pp. 85–100, Springer-Verlag, Berlin, 1977.
5. J. Duchon, Sur l'erreur d'interpolation des fonctions de plusieurs variables par les D^m-splines, *RAIRO Anal. Numer.* **12** (1978), 325–334.

6. M. Golomb and H. F. Weinberger, Optimal approximations and error bounds, *in* "On Numerical Approximations" (R. E. Langer, Ed.), pp. 117–190, Univ. of Wisconsin Press, Madison, 1959.
7. K. Jetter, S. D. Riemenschneider, and Z. Shen, Hermite interpolation on the lattice grid, *SIAM J. Math. Anal.* **25** (1994), 962–975.
8. P.-J. Laurent, "Approximation et Optimasation," Hermann, Paris, 1972.
9. Z. Luo and J. Levesley, "Error Estimates and Convergence Rates for Variational Hermite Interpolation," Research Report 1997/6, Department of Mathematics and Computer Science, University of Leicester, Leicester LE1 7RH, UK.
10. W. Light and H. Wayne, Error estimates for approximation by radial basis functions, *in* "Approximation Theory, Wavelets and Applications" (S. P. Singh, Ed.), pp. 215–246, Kluwer Academic, Dordrecht/Norwell, MA, 1994.
11. W. R. Madych and S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, *Approx. Theory Appl.* **4** (1988), 77–89.
12. W. R. Madych and S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, II, *Math. Comp.* **54** (1990), 211–230.
13. J. Meinguet, Multivariate interpolation at arbitrary points made simple, *Z. Angew. Math. Phys.* **30** (1979), 292–304.
14. F. J. Narcowich and J. D. Ward, Generalized Hermite interpolation via matrix-valued conditionally positive definite functions, *Math. Comp.* (1995).
15. M. J. D. Powell, "The Uniform Convergence of Thin Plate Spline Interpolation in Two Dimensions," Report No. DAMTP 1993/NA16, University of Cambridge, 1993.
16. Xingping Sun, Scattered Hermite interpolation using radial basis functions, *Linear Algebra Appl.* **207** (1994), 135–146.
17. Z. Wu, Hermite–Birkhoff interpolation of scattered data by radial basis functions, *Approx. Theory Appl.* **8** (1992), 1–10.
18. Z. Wu and R. Schabach, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993), 13–27.